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Inequalities for M-matrices and inverse M-matrices ☆

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Abstract

In this paper, we establish some determinantal inequalities concerning M-matrices and inverse M-matrices. The main results are as follows:

1. If $A = (a_{ij})$ is either an $n \times n$ M-matrix or inverse M-matrix , then for any permutation $i_1, i_2, \ldots, i_n$ of $\{1,2,\ldots, n\}$,
   (a) $\det A \leqslant (\prod_{i=1}^n a_{ii}) \prod_{s=2}^n \left(1 - \frac{|a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}|}{a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_s i_s}}\right)$.
   (b) $\det A = \prod_{i=1}^n a_{ii}$ if and only if A is essentially triangular.
2. If $A = (a_{ij})$ is an $n \times n$ M-matrix, $B = (b_{ij})$ is an $n \times n$ inverse M-matrix , $A \circ B$ denotes the Hadamard product of A and B, then $A \circ B$ is an M-matrix, and for any permutation $i_1, i_2, \ldots, i_n$ of $\{1, 2, \ldots, n\}$,

$$\det(A \circ B) \geqslant \det(AB) \prod_{s=2}^n \left(\frac{a_{i_s i_s} \det A[i_1, i_2, \ldots, i_{s-1}]}{\det A[i_1, \ldots, i_{s-1}, i_s]} + \frac{b_{i_s i_s} \det B[i_1, i_2, \ldots, i_{s-1}]}{\det B[i_1, \ldots, i_{s-1}, i_s]} - 1 \right).$$

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1. Introduction

A real matrix is called nonnegative if every entry is nonnegative. For two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \geqslant B$ (Perron–Frobenius order) means $A - B$ is nonnegative, the Hadamard product of A and B is defined and denoted by $A \circ B = (a_{ij}b_{ij})$.

For a positive integer n, let $N = \{1, 2, \ldots, n\}$ throughout. To avoid triviality we always assume that $n > 1$.

Given an $n \times n$ matrix A and a nonempty index set $\alpha = \{i_1, i_2, \ldots, i_s\} \subseteq N$, we will write the principal submatrix of A in rows and columns $i_1, i_2, \ldots, i_s$ as $A[i_1, i_2, \ldots, i_s]$ or $A[\alpha]$. In particular, we set $A(k) = A[N\backslash\{k\}]$, $A_k = A[1, 2, \ldots, k]$ for $k \in N$. We of course adopt the convention that $A[\phi] = 1$.

Let us recall some definitions as follows.

A complex $n \times n$ matrix $A = (a_{ij})$ is called a W-matrix if for any indices $i_1, i_2, \ldots, i_s (s \geqslant 2)$ different from each other in N

$$|a_{i_1i_1}a_{i_2i_2}\cdots a_{i_si_s}| > |a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_si_1}|,$$

and is denoted by $A \in W_n$. Notice that a matrix of order 1 is a W-matrix if and only if it is nonzero.

An $n \times n$ real matrix A is called a Z-matrix if all of its off-diagonal entries are nonpositive, and is denoted by $A \in Z_n$. A Z-matrix is called an M-matrix if it is nonsingular and its inverse is a nonnegative matrix, and denote by M_n the class of all $n \times n$ M-matrices. The class of all matrices whose inverse belongs to M_n, so-called inverse M-matrices, will be denoted by M_n^{-1}.

For convenience, we introduce the following definition:

Definition 1.1. A complex $n \times n$ matrix is called an HF-matrix if the following conditions are satisfied:

(1) All of the principal minors of A are positive.
(2) For arbitrary index sets $\alpha, \beta \subseteq N$, the Hadamard–Fischer inequalities hold, that is

$$\det A[\alpha \cup \beta] \leqslant \det A[\alpha] \det A[\beta] / \det A[\alpha \cap \beta].$$

It is well known that the M-matrices, the inverse M-matrices, the positive definite Hermitian matrices, the totally positive matrices are all HF-matrices, which can be found in [1,2].

For an $n \times n$ positive semi-definite Hermitian matrix $A = (a_{ij})$, Hadamard's inequality states that

$$\det A \leqslant \prod_{i=1}^{n} a_{ii}.$$

Furthermore, equality holds if and only if A is diagonal or A has a zero row or column.

In [3], Zhang and Yang have improved Hadamard's inequality for totally nonnegative and totally positive matrices, and investigate the necessary and sufficient conditions for equality to hold.

Oppenheim's inequality [4, p. 480]: If $A = (a_{ij})$ and $B = (b_{ij})$ are both positive semi-definite Hermitian matrices of order n, then

$$\det(A \circ B) \geqslant \left(\prod_{i=1}^{n} a_{ii}\right) \det B.$$

Oppenheim's inequality has been studied much in the literature. One of the most important results is of course that if $A = (a_{ij})$ and $B = (b_{ij})$ are both M-matrices of order n,then

$$\det(A \circ B) + \det A \cdot \det B \geqslant (\det A)\prod_{i=1}^{n} b_{ii} + (\det B)\prod_{i=1}^{n} a_{ii}, \tag{1}$$

which is attributed to Ando [5]. Notice that (1) also holds for the case that both A and B are positive definite Hermitian matrices of order n [4, Problem 5, p. 483].

Besides there are other developments, for example, Liu and Zhu [6] have improved Oppenheim's inequality for the case that A is an M-matrix and B is either an M-matrix or a positive definite real symmetric matrix; Yang and Liu [7] have strengthened Oppenheim's inequality for the case that both A and B are M-matrices.

In [8], we have strengthened (1) as follows: if both $A = (a_{ij})$ and $B = (b_{ij})$ are M-matrices or positive definite real symmetric matrices of order n, A_k and B_k $(k = 1, 2, \ldots, n)$ are the $k \times k$ leading principal submatrices of A and B, respectively, then

$$\det(A \circ B) \geqslant \det(AB)\prod_{k=2}^{n}\left(\frac{a_{kk}\det A_{k-1}}{\det A_k} + \frac{b_{kk}\det B_{k-1}}{\det B_k} - 1\right). \tag{2}$$

For an $n \times n$ M-matrix $A = (a_{ij})$ and another $n \times n$ inverse M-matrix $B = (b_{ij})$, there exists an analog of Oppenheim's inequality [9, Problem 5, p. 378]:

$$\det(A \circ B) \geqslant \left(\prod_{i=1}^{n} b_{ii}\right)\det A \geqslant \det(AB).$$

It is natural to ask whether Ando's inequality (1) is also valid for $A \in M_n$ and $B \in M_n^{-1}$.

In this paper, we indicate that the answer is affirmative in a stronger form. This means that if $A = (a_{ij})$ is an M-matrix of order n, and $B = (b_{ij})$ is an inverse M-matrix of order n, then the inequality (2) is also valid . On the other hand, lower and upper bounds for the determinant of the Hadamard product of an M-matrix and another inverse M-matrix with the same size are derived.

2. Some lemmas

In this section, we give some lemmas which will be used in the proof of the main results.

M-matrices have important applications, for instance, in iterative methods in numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming.

M-matrices have many equivalent definitions and important properties, but for our purpose, we need only the following Lemmas 2.1 and 2.2, which can be found in [9,10].

Lemma 2.1. *If $A \in Z_n$, then the following statements are equivalent.*

(a) *A is an M-matrix.*
(b) *All of the principal minors of A are positive.*
(c) *All the leading principal minors of A are positive.*

Lemma 2.2. *If $A \in M_n$, $B \in Z_n$ and $B \geqslant A$, then*

(a) *B is an M-matrix.*
(b) $A^{-1} \geqslant B^{-1} \geqslant 0$.
(c) $\det B \geqslant \det A$.

Lemma 2.3. *If* $A = (a_{ij}) \in M_n$, $B = (b_{ij}) \in Z_n$ *and* $B \geqslant A$, *then*

$$\frac{\det B}{b_{kk} \det B(k)} \geqslant \frac{\det A}{a_{kk} \det A(k)} \quad \forall k \in N.$$

Proof. Write $A^{-1} = (\alpha_{ij})$ and $B^{-1} = (\beta_{ij})$. By Lemma 2.2, $A^{-1} \geqslant B^{-1} \geqslant 0$, hence

$$b_{kk}\beta_{kk} = \sum_{i \neq k} |b_{ki}|\beta_{ik} + 1 \leqslant \sum_{i \neq k} |a_{ki}|\alpha_{ik} + 1 = a_{kk}\alpha_{kk}.$$

Since $\beta_{kk} = \det B(k)/\det B$, $\alpha_{kk} = \det A(k)/\det A$, we have

$$0 < \frac{b_{kk} \det B(k)}{\det B} \leqslant \frac{a_{kk} \det A(k)}{\det A}.$$

Therefore

$$\frac{\det B}{b_{kk} \det B(k)} \geqslant \frac{\det A}{a_{kk} \det A(k)}. \qquad \square$$

Lemma 2.4. *Suppose a real* $n \times n$ *matrix* $A = (a_{ij})$ *is partitioned as* $A = \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix}$. *If* $\det A_{n-1} > 0$, x *is a real number*, *then*

$$\det \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & x \end{pmatrix} > 0 \text{ if and only if } x > a_{nn} - \det A/\det A_{n-1}.$$

Proof. This can be found in [8, Lemma 2.1(a)]. $\square$

Lemma 2.5. $\forall \varepsilon > 0$. *If* $A = \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix} \in M_n$, *then*

$$B = \begin{pmatrix} A_{n-1} & A_{12} \\ A_{21} & a_{nn} - \det A/\det A_{n-1} + \varepsilon \end{pmatrix} \in M_n.$$

Proof. Since $A_{n-1} \in M_{n-1}$, $\det B = \varepsilon \det A_{n-1} > 0$, all the leading principal minors of B are positive by Lemma 2.1, thus $B \in M_n$. $\square$

Lemma 2.6. *If* $A \in M_n$, $B \in W_n$, *and* $B \geqslant 0$, *then* $A \circ B \in M_n$.

Proof. Observe that $A \circ B \in Z_n$, this is a direct consequence of Theorem (3.1) of [11]. $\square$

Lemma 2.7. *If* $B = (b_{ij}) \in M_n^{-1} (n \geqslant 3)$, *then for any indices* i, j, k *in* N

$$0 \leqslant b_{ik}b_{kj} \leqslant b_{kk}b_{ij}.$$

Proof. This follows from [12, Lemma 2.2(ii)]. $\square$

Lemma 2.8. *Let* $A \in M_n$, $B \in M_n^{-1}$. *If* P *is a permutation matrix of order* n, *then* $P^{-1}AP \in M_n$, $P^{-1}BP \in M_n^{-1}$, *and*

$$\det[(P^{-1}AP) \circ (P^{-1}BP)] = \det(A \circ B). \tag{3}$$

Proof. It is quite evident that $P^{-1}AP \in M_n$ and $P^{-1}BP \in M_n^{-1}$.
Since $(P^{-1}AP) \circ (P^{-1}BP) = P^{-1}(A \circ B)P$, so (3) holds. $\square$

3. Main results

In this section, we state and prove our main results.

Theorem 3.1. *If* $A = (a_{ij}) \in M_n \cup M_n^{-1}$, *then for any* $\alpha = \{i_1, i_2, \ldots, i_s\} \subseteq N$, *where* $i_1, i_2, \ldots, i_s$ *are mutually distinct*,

$$\frac{\det A}{a_{i_s i_s} \det A(i_s)} \leqslant 1 - \frac{|a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}|}{a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_s i_s}}. \tag{4}$$

Proof. Let us distinguish two cases:

Case 1. $A \in M_n$. Put $S = \{(i_1, i_2), \ldots, (i_{s-1} i_s), (i_s, i_1), (1, 1,), (2, 2), \ldots, (n, n)\}$.

We define an $n \times n$ matrix $B = (b_{ij})$ in the following manner:
$b_{ij} = a_{ij}$ if $(i, j) \in S$; $b_{ij} = 0$ if $(i, j) \notin S$.
Obviously, $B \geqslant A$, by Lemma 2.3, we have

$$\begin{aligned}\frac{\det A}{a_{i_s i_s} \det A(i_s)} &\leqslant \frac{\det B}{b_{i_s i_s} \det B(i_s)} \\ &= \left(\prod_{i \in \alpha} a_{ii} - |a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}|\right) \cdot \prod_{i \notin \alpha} a_{ii} \Big/ \prod_{i=1}^{n} a_{ii} \\ &= 1 - |a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1}| \Big/ \prod_{i \in \alpha} a_{ii}.\end{aligned}$$

Case 2. $A \in M_n^{-1}$. First, let us prove the following inequality by induction on n, the order of matrices.

$$\frac{\det A}{a_{i_n i_n} \det A(i_n)} \leqslant 1 - \frac{a_{i_1 i_2} \cdots a_{i_{n-1} i_n} a_{i_n i_1}}{a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_n i_n}}. \tag{5}$$

One can easily verify that (5) holds with equality for $n = 2$.

Now we assume that $n > 2$, and the inequality (5) is true in case the order of matrices is $n - 1$.
Observe that $0 \leqslant a_{i_n i_1} a_{i_1 i_2} \leqslant a_{i_1 i_1} a_{i_n i_2}$ by Lemma 2.7, since A is an HF-matrix, we have

$$\det A \leqslant \frac{\det A[i_1, i_2, \ldots, i_{n-1}] \det A[i_2, i_3, \ldots, i_n]}{\det A[i_2, i_3, \ldots, i_{n-1}]},$$

whence

$$\begin{aligned}\frac{\det A}{a_{i_n i_n} \det A(i_n)} &\leqslant \frac{\det A[i_2, i_3, \ldots, i_n]}{a_{i_n i_n} \det A[i_2, i_3, \ldots, i_{n-1}]} \\ &\leqslant 1 - \frac{a_{i_2 i_3} \ldots a_{i_{n-1} i_n} a_{i_n i_2}}{a_{i_2 i_2} \ldots a_{i_{n-1} i_{n-1}} a_{i_n i_n}} \quad \text{(by the induction hypothesis)}\end{aligned}$$

$$
\begin{aligned}
&= 1 - \frac{a_{i_2i_3}\cdots a_{i_{n-1}i_n}(a_{i_ni_2}a_{i_1i_1})}{a_{i_1i_1}a_{i_2i_2}\cdots a_{i_ni_n}} \\
&\leqslant 1 - \frac{a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{n-1}i_n}a_{i_ni_1}}{a_{i_1i_1}a_{i_2i_2}\cdots a_{i_ni_n}}.
\end{aligned}
$$

Therefore, (5) is proved.

Again apply Hadamard–Fischer inequality, we can get

$$
\begin{aligned}
\frac{\det A}{a_{i_si_s}\det A(i_s)} &\leqslant \frac{\det A[i_1, i_2, \ldots, i_{s-1}, i_s]}{a_{i_si_s}\det A[i_1, i_2, \ldots, i_{s-1}]} \\
&\leqslant 1 - \frac{a_{i_1i_2}\ldots a_{i_{s-1}i_s}a_{i_si_1}}{a_{i_1i_1}a_{i_2i_2}\ldots a_{i_si_s}} \text{ (by the inequality (5))}.
\end{aligned}
$$

This completes the proof. □

Corollary 3.1. *Let $B = (b_{ij}) \in M_n^{-1}$, then*

(a) *All of the principal submatrices of B are W-matrices.*
(b) $\forall \varepsilon > 0$. *If B is partitioned as $B = \begin{pmatrix} B_{n-1} & B_{12} \\ B_{12} & b_{nn} \end{pmatrix}$, then*

$$
\begin{pmatrix} B_{n-1} & B_{12} \\ B_{12} & b_{nn} - \det B/\det B_{n-1} + \varepsilon \end{pmatrix} \in W_n.
$$

Proof. Let $i_1, i_2, \ldots, i_s \in N$ be s distinct indices, $s \geqslant 2$. Theorem 3.1 yields

$$
0 < \frac{\det B}{b_{i_si_s}\det B(i_s)} \leqslant 1 - \frac{b_{i_1i_2}\cdots b_{i_{s-1}i_s}b_{i_si_1}}{b_{i_1i_1}b_{i_2i_2}\cdots b_{i_si_s}}. \tag{6}
$$

Hence

$$
b_{i_1i_1}b_{i_2i_2}\cdots b_{i_si_s} > b_{i_1i_2}\cdots b_{i_{s_1}i_s}b_{i_si_1} \geqslant 0,
$$

which proves (a).

Now we take $i_s = n$. According to (6), we can easily obtain

$$
b_{i_1i_1}b_{i_2i_2}\cdots b_{i_{s-1}i_{s-1}}(b_{nn} - \det B/\det B_{n-1}) \geqslant b_{i_1i_2}\cdots b_{i_{s-1}i_s}b_{i_si_1} \geqslant 0.
$$

Therefore

$$
b_{i_1i_1}b_{i_2i_2}\cdots b_{i_{s-1}i_{s-1}}(b_{nn} - \det B/\det B_{n-1} + \varepsilon) > b_{i_1i_2}\cdots b_{i_{s-1}i_s}b_{i_si_1} \geqslant 0.
$$

By the definition of W-matrix, we claim that (b) is valid. □

Theorem 3.2. *If $A = (a_{ij}) \in M_n \cup M_n^{-1}$, then for any permutation $i_1, i_2, \ldots, i_n$ of N,*

(a) $\det A \leqslant \left(\prod_{i=1}^n a_{ii}\right)\prod_{s=2}^n\left(1 - \frac{|a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_si_1}|}{a_{i_1i_1}a_{i_2i_2}\cdots a_{i_si_s}}\right)$.
(b) $\det A = \prod_{i=1}^n a_{ii}$ *if and only if A is essentially triangular.*

Proof. For $s = 2, 3, \ldots, n$, we deduce by Theorem 3.1 that

$$
\frac{\det A[i_1, \ldots, i_{s-1}i_s]}{a_{i_si_s}\det A[i_1, \ldots, i_{s-1}]} \leqslant 1 - \frac{|a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_si_1}|}{a_{i_1i_1}a_{i_2i_2}\cdots a_{i_si_s}}.
$$

By multiplying these inequalities, we obtain

$$\prod_{s=2}^{n}\frac{\det A[i_1,\ldots,i_{s-1}i_s]}{a_{i_si_s}\det A[i_1,\ldots,i_{s-1}]}\leqslant\prod_{s=2}^{n}\left(1-\frac{|a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_si_1}|}{a_{i_1i_1}a_{i_2i_2}\cdots a_{i_si_s}}\right),$$

which yields

$$\det A\leqslant\left(\prod_{i=1}^{n}a_{ii}\right)\prod_{s=2}^{n}\left(1-\frac{|a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_si_1}|}{a_{i_1i_1}a_{i_2i_2}\cdots a_{i_si_s}}\right). \tag{7}$$

Recall that a square matrix B is called essentially triangular if PBP^{-1} is triangular for some permutation matrix P. Using Frobenius normal form of A [13, Theorem 3.2.4], it follows that A is essentially triangular if and only if for any indices $i_1, i_2, \ldots, i_s (s \geqslant 2)$ different from each other in N

$$a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_si_1}=0.$$

By (7), we claim that $\det A=\prod_{i=1}^{n}a_{ii}$ holds if and only if A is essentially triangular. This completes the proof. □

Below we establish lower and upper bounds for the determinant of the Hadamard product of an M-matrix and another inverse M-matrix with the same size.

Theorem 3.3. *If $A=(a_{ij})\in M_n$, $B=(b_{ij})\in M_n^{-1}$, then $A\circ B\in M_n$, and for any permutation $i_1, i_2, \ldots, i_n$ of N,*

$$\det(A\circ B)\geqslant\det(AB)\prod_{s=2}^{n}\left(\frac{a_{i_si_s}\det A[i_1,\ldots,i_{s-1}]}{\det A[i_1,\ldots,i_{s-1},i_s]}+\frac{b_{i_si_s}\det B[i_1,\ldots,i_{s-1}]}{\det B[i_1,\ldots,i_{s-1},i_s]}-1\right), \tag{8}$$

and

$$\det(A\circ B)\leqslant\left(\prod_{i=1}^{n}a_{ii}b_{ii}\right)\prod_{s=2}^{n}\left(1-\frac{|a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_si_1}|}{a_{i_1i_1}a_{i_2i_2}\cdots a_{i_si_s}}\cdot\frac{b_{i_1i_2}\cdots b_{i_{s-1}i_s}b_{i_si_1}}{b_{i_1i_1}b_{i_2i_2}\cdots b_{i_si_s}}\right). \tag{9}$$

Proof. $\forall k\in N$, we have $A_k\in M_k$ and $B_k\in W_k$ by Corollary 3.1(a). Since $B_k\geqslant 0$, Lemma 2.6 yields that $A_k\circ B_k\in M_k$.

To prove (8), according to Lemma 2.8, we need only to prove the following inequality:

$$\det(A\circ B)\geqslant\det(AB)\prod_{k=2}^{n}\left(\frac{a_{kk}\det A_{k-1}}{\det A_k}+\frac{b_{kk}\det B_{k-1}}{\det B_k}-1\right). \tag{10}$$

In fact, it is not difficult to verify that (10) holds with equality for $n=2$. Now we assume that $n>2$. For $k=2,3,\ldots,n$, we partition A_k and B_k as

$$A_k=\begin{pmatrix}A_{k-1} & A_{12}^{(k)}\\ A_{21}^{(k)} & a_{kk}\end{pmatrix},\quad B_k=\begin{pmatrix}B_{k-1} & B_{12}^{(k)}\\ B_{21}^{(k)} & b_{kk}\end{pmatrix}.$$

$\forall\varepsilon>0$, Lemma 2.5, Corollary 3.1(b) and Lemma 2.6 imply that

$$\begin{pmatrix}A_{k-1} & A_{12}^{(k)}\\ A_{21}^{(k)} & a_{kk}-\det A_k/\det A_{k-1}+\varepsilon\end{pmatrix}\circ\begin{pmatrix}B_{k-1} & B_{12}^{(k)}\\ B_{21}^{(k)} & b_{kk}-\det B_k/\det B_{k-1}+\varepsilon\end{pmatrix}\in M_k.$$

By Lemma 2.4, we have

$$\left(a_{kk} - \frac{\det A_k}{\det A_{k-1}} + \varepsilon\right)\left(b_{kk} - \frac{\det B_k}{\det B_{k-1}} + \varepsilon\right) > a_{kk}b_{kk} - \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})}.$$

Letting $\varepsilon \to 0$, we obtain

$$\left(a_{kk} - \frac{\det A_k}{\det A_{k-1}}\right)\left(b_{kk} - \frac{\det B_k}{\det B_{k-1}}\right) \geqslant a_{kk}b_{kk} - \frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})}.$$

From this we can get

$$\frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})} \geqslant \frac{\det(A_k B_k)}{\det(A_{k-1} B_{k-1})}\left(\frac{a_{kk}\det A_{k-1}}{\det A_k} + \frac{b_{kk}\det B_{k-1}}{\det B_k} - 1\right).$$

Multiplying these inequalities

$$\prod_{k=2}^{n}\frac{\det(A_k \circ B_k)}{\det(A_{k-1} \circ B_{k-1})} \geqslant \prod_{k=2}^{n}\frac{\det(A_k B_k)}{\det(A_{k-1} B_{k-1})}\left(\frac{a_{kk}\det A_{k-1}}{\det A_k} + \frac{b_{kk}\det B_{k-1}}{\det B_k} - 1\right).$$

This means that (10) is valid.

Taking into account that $A \circ B \in M_n$, (9) is an immediate consequence of Theorem 3.2(a). The proof is complete. □

Corollary 3.2. *If $A = (a_{ij}) \in M_n$, $B = (b_{ij}) \in M_n^{-1}$, then*

$$\det(A \circ B) + \det(AB) \geqslant \det A \prod_{i=1}^{n} b_{ii} + \det B \prod_{i=1}^{n} a_{ii}. \tag{11}$$

Proof. Obviously, (11) is equivalent to

$$\det(A \circ B) \geqslant \det(AB)\left(\frac{\prod_{i=1}^{n} a_{ii}}{\det A} + \frac{\prod_{i=1}^{n} b_{ii}}{\det B} - 1\right). \tag{12}$$

By the inequality (8), (12) follows from the following inequality:

If both $A = (a_{ij})$ and $B = (b_{ij})$ are HF-matrices of order n, then

$$\prod_{k=2}^{n}\left(\frac{a_{kk}\det A_{k-1}}{\det A_k} + \frac{b_{kk}\det B_{k-1}}{\det B_k} - 1\right) \geqslant \frac{\prod_{i=1}^{n} a_{ii}}{\det A} + \frac{\prod_{i=1}^{n} b_{ii}}{\det B} - 1 + \varepsilon_n(A, B), \tag{13}$$

where

$$\varepsilon_n(A, B) = \sum_{k=2}^{n}\left[\left(\frac{a_{kk}\det A_{k-1}}{\det A_k} - 1\right)\left(\frac{\prod_{i=1}^{k-1} b_{ii}}{\det B_{k-1}} - 1\right) + \left(\frac{b_{kk}\det B_{k-1}}{\det B_k} - 1\right)\left(\frac{\prod_{i=1}^{k-1} a_{ii}}{\det A_{k-1}} - 1\right)\right] \geqslant 0.$$

We prove it by induction on n. It is easy to see that (13) is true with equality for $n = 2$.

Now assume that $n > 2$ and (13) is true for the case $n - 1$, then the induction hypothesis and our assumption yield the chain of inequalities

$$
\begin{aligned}
&\prod_{k=2}^{n}\left(\frac{a_{kk}\det A_{k-1}}{\det A_k}+\frac{b_{kk}\det B_{k-1}}{\det B_k}-1\right)\\
&\quad\geqslant\left(\frac{a_{nn}\det A_{n-1}}{\det A}+\frac{b_{nn}\det B_{n-1}}{\det B}-1\right)\\
&\quad\quad\times\left[\left(\frac{\prod_{i=1}^{n-1}a_{ii}}{\det A_{n-1}}+\frac{\prod_{i=1}^{n-1}b_{ii}}{\det B_{n-1}}-1\right)+\varepsilon_{n-1}(A_{n-1},B_{n-1})\right]\\
&\quad\geqslant\left(\frac{a_{nn}\det A_{n-1}}{\det A}+\frac{b_{nn}\det B_{n-1}}{\det B}-1\right)\left(\frac{\prod_{i=1}^{n-1}a_{ii}}{\det A_{n-1}}+\frac{\prod_{i=1}^{n-1}b_{ii}}{\det B_{n-1}}-1\right)+\varepsilon_{n-1}(A_{n-1},B_{n-1})\\
&\quad=\frac{\prod_{i=1}^{n}a_{ii}}{\det A}+\frac{\prod_{i=1}^{n}b_{ii}}{\det B}+\frac{\prod_{i=1}^{n-1}b_{ii}}{\det B_{n-1}}\left(\frac{a_{nn}\det A_{n-1}}{\det A}-1\right)+\frac{\prod_{i=1}^{n-1}a_{ii}}{\det A_{n-1}}\left(\frac{b_{nn}\det B_{n-1}}{\det B}-1\right)\\
&\quad\quad-\left[\left(\frac{a_{nn}\det A_{n-1}}{\det A}-1\right)+\left(\frac{b_{nn}\det B_{n-1}}{\det B}-1\right)+1\right]+\varepsilon_{n-1}(A_{n-1},B_{n-1})\\
&\quad=\frac{\prod_{i=1}^{n}a_{ii}}{\det A}+\frac{\prod_{i=1}^{n}b_{ii}}{\det B}-1+\left(\frac{a_{nn}\det A_{n-1}}{\det A}-1\right)\left(\frac{\prod_{i=1}^{n-1}b_{ii}}{\det B_{n-1}}-1\right)\\
&\quad\quad+\left(\frac{b_{nn}\det B_{n-1}}{\det B}-1\right)\left(\frac{\prod_{i=1}^{n-1}a_{ii}}{\det A_{n-1}}-1\right)+\varepsilon_{n-1}(A_{n-1},B_{n-1})\\
&\quad=\frac{\prod_{i=1}^{n}a_{ii}}{\det A}+\frac{\prod_{i=1}^{n}b_{ii}}{\det B}-1+\varepsilon_n(A,B).
\end{aligned}
$$

This completes the induction. □

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